

BOUNDS FOR HILBERT COEFFICIENTS

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ABSTRACT. We compute the Hilbert coefficients of a graded module with pure resolution and discuss lower and upper bounds for these coefficients for arbitrary graded modules.

INTRODUCTION

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables, and let N be any graded S -module of dimension d . Then for $i \gg 0$, the numerical function $H(N, i) = \sum_{j \leq i} \dim_K N_j$ is a polynomial function of degree d , see [1, 4.1.6]. In other words, there exists a polynomial $P_N(x) \in \mathbb{Z}[x]$ such that

$$H(N, i) = P_N(i) \quad \text{for all } i \gg 0.$$

The polynomial $P_N(x)$ is called the *Hilbert polynomial* of N . It can be written in the form

$$P_N(x) = \sum_{i=0}^d (-1)^i e_i(N) \binom{x+d-i}{d-i}$$

with integer coefficients $e_i(N)$, called the *Hilbert coefficients* of N .

In the first section we will give explicit formulas for the $e_i(N)$ in case N has a pure resolution. In the second section we use this result and a conjecture of Boij and Söderberg [2] to get conjectural lower and upper bounds for the Hilbert coefficients. We also discuss a few cases for which these bounds hold. These bounds generalize the conjectured bounds for the multiplicity, due to Huneke, Srinivasan and the first author of this paper, see [4]. A rather complete survey of the multiplicity conjecture can be found in [5]. For more recent results we refer to [8], [7], [9] and [6].

1. THE HILBERT COEFFICIENTS OF A MODULE WITH PURE RESOLUTION

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables, and let N be a finitely generated graded S -module. We say N has a pure resolution of type

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(d_0, d_1, \dots, d_s) if its minimal graded free S -resolution is of the form

$$0 \longrightarrow S^{\beta_s}(-d_s) \longrightarrow \cdots \longrightarrow S^{\beta_1}(-d_1) \longrightarrow S^{\beta_0}(-d_0) \longrightarrow 0.$$

The main result of this section is

Theorem 1.1. *Let N be a finitely generated graded Cohen-Macaulay S -module of codimension s with pure resolution of type (d_0, d_1, \dots, d_s) with $d_0 = 0$. Then the Hilbert coefficients of N are*

$$e_i(N) = \beta_0 \frac{\prod_{j=1}^s d_j}{(s+i)!} \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_i \leq s} \prod_{k=1}^i (d_{j_k} - (j_k + k - 1)), \quad i = 0, \dots, n-s.$$

Proof. We first recall a few facts about Hilbert series and multiplicities as described in [1]. The Hilbert series $H_N(t) = \sum_i H(N, i)t^i$ is a rational function of the form

$$H_N(t) = \frac{Q_N(t)}{(1-t)^{d+1}},$$

where $d = n - s$ is the dimension of N . The Hilbert coefficients $e_i = e_i(N)$ of N can be computed according to the formula

$$e_i = \frac{Q_N^{(i)}(1)}{i!}, \quad i = 0, \dots, d.$$

On the other hand by using the additivity of Hilbert functions, the free resolution of N yields the presentation

$$H_N(t) = \frac{P_N(t)}{(1-t)^{n+1}} \quad \text{with} \quad P_N(t) = \sum_{j=0}^s (-1)^j \beta_j t^{d_j}.$$

Thus we see that $P_N(t) = Q_N(t)(1-t)^s$. This yields

$$(1) \quad e_i = (-1)^s \frac{P_N^{(s+i)}(1)}{(s+i)!}, \quad i = 0, \dots, d.$$

For any two integers $0 \leq a \leq b$ we set

$$g_a(b) = \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_a \leq b \\ 2}} i_1 i_2 \cdots i_a.$$

Then we have

$$\begin{aligned}
P_N^{(s+i)}(1) &= \sum_{j=0}^s (-1)^j \beta_j \prod_{k=0}^{s+i-1} (d_j - k) \\
&= \sum_{j=0}^s (-1)^j \beta_j \sum_{k=0}^{s+i} (-1)^{s+i-k} g_{s+i-k} (s+i-1) d_j^k \\
&= \sum_{k=0}^{s+i} (-1)^{s+i-k} g_{s+i-k} (s+i-1) \sum_{j=0}^s (-1)^j \beta_j d_j^k.
\end{aligned}$$

Hence if we set $a_k = \sum_{j=0}^s (-1)^j \beta_j d_j^{k+s}$ for all $k \geq 0$ and observe that $\sum_{j=0}^s (-1)^j \beta_j d_j^k = 0$ for all $k < s$ (see [4] where the proof of this fact is given in the cyclic case), we obtain together with (1) the following identities

$$(2) \quad (-1)^s (s+i)! e_i = \sum_{k=0}^i (-1)^{i-k} g_{i-k} (s+i-1) a_k, \quad i = 0, \dots, d.$$

In order to compute the a_i we consider for each i the following matrix

$$B_i = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_s d_s \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_s d_s^2 \\ \vdots & \vdots & & \vdots \\ \beta_1 d_1^{s-1} & \beta_2 d_2^{s-1} & \cdots & \beta_s d_s^{s-1} \\ \beta_1 d_1^{s+i} & \beta_2 d_2^{s+i} & \cdots & \beta_s d_s^{s+i} \end{pmatrix}.$$

Replacing the last column of B_i by the alternating sum of its columns we obtain the matrix B'_i for which $\det B'_i = (-1)^s \det B_i$ and whose last column is the transpose of $(0, 0, \dots, a_i)$. It follows that

$$(3) \quad a_i = (-1)^s \det B_i / \det C,$$

where

$$C = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_{s-1} d_{s-1} \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_{s-1} d_{s-1}^2 \\ \vdots & \vdots & & \vdots \\ \beta_1 d_1^{s-1} & \beta_2 d_2^{s-1} & \cdots & \beta_{s-1} d_{s-1}^{s-1} \end{pmatrix}.$$

Note that $\det N = \beta_1 \cdots \beta_{s-1} d_1 \cdots d_{s-1} \det V(d_1, \dots, d_{s-1})$, where $V(d_1, \dots, d_{s-1})$ is the Vandermonde matrix for the sequence d_1, d_2, \dots, d_{s-1} . Hence we obtain

$$\det C = \beta_1 \cdots \beta_{s-1} d_1 \cdots d_{s-1} \prod_{1 \leq i < j \leq s-1} (d_j - d_i).$$

On the other hand we have

$$\det B_i = \beta_1 \cdots \beta_s d_1 \cdots d_s \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_s \\ \vdots & \vdots & & \vdots \\ d_1^{s-2} & d_2^{s-2} & \cdots & d_s^{s-2} \\ d_1^{s+i-1} & d_2^{s+i-1} & \cdots & d_s^{s+i-1} \end{pmatrix}.$$

According to the subsequent Lemma 1.2 we have

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_s \\ \vdots & \vdots & & \vdots \\ d_1^{s-2} & d_2^{s-2} & \cdots & d_s^{s-2} \\ d_1^{s+i-1} & d_2^{s+i-1} & \cdots & d_s^{s+i-1} \end{pmatrix} = f_i(d_1, \dots, d_s) \cdot \prod_{1 \leq j < k \leq s} (d_k - d_j),$$

where for each integer $k \geq 0$ we set

$$f_k(g_1, \dots, g_s) = \sum g_1^{c_1} \cdots g_s^{c_s}.$$

Here the sum is taken over all integer vectors $c = (c_1, \dots, c_s)$ with $c_i \geq 0$ for all i and $|c| = \sum_{i=1}^s c_i = k$.

Thus by (3) we have

$$a_i = (-1)^s \beta_s d_s f_i(d_1, \dots, d_s) \prod_{j=1}^{s-1} (d_s - d_j).$$

Now we use that fact that $\beta_s = \beta_0 \prod_{j=1}^{s-1} d_j / \prod_{j=1}^{s-1} (d_s - d_j)$ (see [3] or [2]) and we obtain

$$a_i = (-1)^s \beta_0 d_1 \cdots d_s f_i(d_1, \dots, d_s).$$

This result together with (2) yields the formulas

$$(4) \quad e_i = \beta_0 \frac{d_1 \cdots d_s}{(s+i)!} \sum_{j=0}^i (-1)^{i-j} g_{i-j}(s+i-1) f_j(d_1, \dots, d_s).$$

Expanding the products in the following sum

$$\sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_i \leq s} \prod_{k=1}^i (d_{j_k} - (j_k + k - 1))$$

yields

$$\sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_i \leq s} \prod_{k=1}^i (d_{j_k} - (j_k + k - 1)) = \sum_{j=0}^i (-1)^{i-j} g_{i-j}(s+i-1) f_j(d_1, \dots, d_s).$$

Hence the desired formulas for the e_i follow from (4). \square

It remains to prove

Lemma 1.2. *For all $k \geq s - 1 \geq 0$ one has*

$$\det \begin{pmatrix} 1 & \dots & 1 \\ d_1 & \dots & d_s \\ \vdots & & \vdots \\ d_1^{s-2} & \dots & d_s^{s-2} \\ d_1^k & \dots & d_s^k \end{pmatrix} = f_{k-s+1}(d_1, \dots, d_s) \cdot \prod_{1 \leq i < j \leq s} (d_j - d_i).$$

Proof. Given integers $1 \leq r \leq s$ and $k \geq s$ we define the matrix

$$A_r^{(k)} = (a_{ij}^{(k)})_{\substack{i=1, \dots, s-r+1 \\ j=1, \dots, s-r+1}}$$

with

$$a_{ij}^{(k)} = \begin{cases} f_{i-1}(d_1, \dots, d_{r-1}, d_{r-1+j}), & \text{for } i \leq s-r, j = 1, \dots, s-r+1 \\ f_{k-r+1}(d_1, \dots, d_{r-1}, d_{r-1+j}), & \text{for } i = s-r+1, j = 1, \dots, s-r+1. \end{cases}$$

Notice that $A_1^{(k)}$ is the matrix whose determinant we want to compute, while $A_s^{(k)}$ is the 1×1 -matrix with entry $f_{k-s+1}(d_1, \dots, d_{s-1}, d_s)$.

Next observe that for each integer $\ell > 0$ and all $j > 1$ one has

$$f_\ell(d_1, \dots, d_{r-1}, d_{r-1+j}) - f_\ell(d_1, \dots, d_{r-1}, d_r) = (d_{r-1+j} - d_r) \cdot f_{\ell-1}(d_1, \dots, d_r, d_{r-1+j}).$$

Hence if subtract the first column from the other columns of $A_r^{(k)}$ and expand then this new matrix with respect to the first row (which is $(1, 0, \dots, 0)$) we see that

$$\det A_r^{(k)} = (d_{r+1} - d_r)(d_{r+2} - d_r) \cdots (d_s - d_r) \det A_{r+1}^{(k)}.$$

Form this we obtain that

$$\det A_1^{(k)} = \det A_s^{(k)} \cdot \prod_{1 \leq i < j \leq s} (d_j - d_i) = f_{k-s+1}(d_1, \dots, d_{s-1}, d_s) \cdot \prod_{1 \leq i < j \leq s} (d_j - d_i),$$

as desired. \square

For $i = 0, 1, 2$ the formulas for the Hilbert coefficients read as follows:

$$e_0(N) = \beta_0 \frac{\prod_{i=1}^s d_i}{s!};$$

$$e_1(N) = \beta_0 \frac{\prod_{i=1}^s d_i}{(s+1)!} \sum_{i=1}^s (d_i - i);$$

$$e_2(N) = \beta_0 \frac{\prod_{i=1}^s d_i}{(s+2)!} \sum_{1 \leq i < j \leq s} (d_i - i)(d_j - j - 1).$$

In the special case that N has a d -linear resolution, our formulas yield

$$e_i(N) = \beta_0 \binom{d+s-1}{s+i} \binom{s+i-1}{i}.$$

Remark 1.3. The assumption made in Theorem 1.1 that d_0 should be zero, is not essential. It is only made to simplify the formulas for the Hilbert coefficients. While for the multiplicity we have $e_0(N) = e_0(N(a))$ for any shift a , the other Hilbert coefficients transform as follows: if N has a pure resolution of type (d_0, d_1, \dots, d_s) , then $N(d_0)$ has pure resolution of type $(0, d_1 - d_0, \dots, d_s - d_0)$ whose Hilbert coefficient we know by Theorem 1.1.

On the other hand we have $P_N(x) = P_{N(d_0)}(x + d_0)$, from which one deduces that

$$e_i(N) = \sum_{j=0}^i \binom{d_0 - 1 + i - j}{d_0 - 1} e_i(N(d_0)) \quad \text{for } i = 0, \dots, n - s.$$

2. UPPER AND LOWER BOUNDS

Given a sequence d_1, d_2, \dots, d_s of integers. We set

$$h_i(d_1, \dots, d_s) = \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_i \leq s} \prod_{k=1}^i (d_{j_k} - (j_k + k - 1))$$

for $i = 0, \dots, n - s$. This definition will simplify notation in the following discussions.

Let N be any finitely generated graded Cohen-Macaulay S -module of projective dimension s and graded Betti numbers β_{ij} . For each $i = 1, \dots, s$, the minimal and maximal shifts of N in homological degree i are defined by $m_i = \min\{j : \beta_{ij} \neq 0\}$ and $M_i = \max\{j : \beta_{ij} \neq 0\}$.

In case N is generated in degree 0 and has a pure resolution of type (d_1, \dots, d_s) , we have $m_i = M_i = d_i$ for all i , and Theorem 1.1 tells us that

$$e_i(N) = \beta_0 \frac{d_1 d_2 \cdots d_s}{(s + i)!} h_i(d_1, \dots, d_s) \quad \text{for } i = 0, 1, \dots, n - s.$$

In analogy to the so-called multiplicity conjecture we now state

Conjecture 2.1. *Let N be a finitely generated graded Cohen-Macaulay S -module of codimension s generated in degree 0. Then*

$$\beta_0 \frac{m_1 m_2 \cdots m_s}{(s + i)!} h_i(m_1, \dots, m_s) \leq e_i(N) \leq \beta_0 \frac{M_1 M_2 \cdots M_s}{(s + i)!} h_i(M_1, \dots, M_s)$$

for $i = 0, 1, \dots, n - s$.

Next we recall a conjecture of Boij and Söderberg [2]: for any strictly increasing sequences of integers $m = (m_0, \dots, m_s)$ and $M = (M_0, \dots, M_s)$, let $V_{m,M}$ be the vector space over the rational numbers of all matrices $(\beta_{i,j})$ such that:

(a) $\beta_{i,j}$ is a solution to the system of linear equations

$$\begin{cases} \sum_{i,j} (-1)^i \beta_{i,j} &= 0 \\ \sum_{i,j} (-1)^i j \beta_{i,j} &= 0 \\ &\vdots \\ \sum_{i,j} (-1)^i j^{s-1} \beta_{i,j} &= 0 \end{cases}$$

(b) $\beta_{i,j} = 0$ whenever $j < m_i$ or $j > M_i$ (or $i < 0$ or $i > s$).

Note that the graded Betti numbers of any graded Cohen-Macaulay module N of codimension s satisfies condition (a). Moreover if the maximal and minimal shifts of N are the numbers m_i and M_i , then this Betti diagram belongs to $V_{m,M}$. The set of Betti diagrams in $V_{m,M}$ is denoted by $B_{m,M}$. It is an additively closed subset of $V_{m,M}$.

To each $(\beta_{ij}) \in B_{m,M}$ we assign the normalized Betti diagram $(\bar{\beta}_{ij}) = (\beta_{ij}/\beta_0)$ and define the subset $\bar{B}_{m,M} = \{(\bar{\beta}_{ij}) : (\beta_{ij}) \in B_{m,M}\}$ of $V_{m,M}$. The set $\bar{B}_{m,M}$ is closed under convex combinations with rational coefficients.

For any strictly increasing sequence of integers $d = (d_0, d_1, \dots, d_s)$, the matrix $\pi(d)$ defined by

$$\pi(d)_{i,j} = \begin{cases} (-1)^{i+1} \prod_{\substack{k \neq i \\ k \neq 0}} \frac{d_k - d_0}{d_k - d_i} & \text{if } j = d_i, \\ 0 & \text{if } j \neq d_i, \end{cases}$$

is called a pure diagram.

For any two strictly increasing sequences of integers $m = (m_0, \dots, m_s)$ and $M = (M_0, \dots, M_s)$ we denote by $\Pi_{m,M}$ the set of all pure diagrams in $V_{m,M}$. Note that $\Pi_{m,M}$ is just the set of pure diagrams $\pi(d)$ with $m_i \leq d_i \leq M_i$ for all i .

Now we have the following

Conjecture 2.2 (Boij, Söderberg). $\bar{B}_{m,M}$ is the convex hull of $\Pi_{m,M}$.

If it happens that N is a graded Cohen-Macaulay module generated in degree 0 with pure resolution of type $d = (d_1, \dots, d_s)$, then the normalized Betti diagram of N is just $\pi(d)$ (with $d_0 = 0$), as follows from [3] (see also [2]). Hence we define for $i = 0, \dots, n - s$, the Hilbert coefficients of a pure diagram $\pi(d)$ for which $d_0 = 0$ as

$$e_i(\pi(d)) = \frac{d_1 d_2 \cdots d_s}{(s+i)!} h_i(d_1, \dots, d_s),$$

no matter whether or not d is the type of a Cohen-Macaulay module with pure resolution.

The following observation justifies our conjecture.

Proposition 2.3. *Conjecture 2.2 implies Conjecture 2.1.*

Proof. Let N is a graded Cohen-Macaulay module of codimension s generated in degree 0, and let D be the normalized Betti diagram of N . Let $m = (m_1, \dots, m_s)$ and

$M = (M_1, \dots, M_s)$ be the sequences of minimal and maximal shifts of D . Assuming Conjecture 2.2 we have

$$D = \sum_{\pi(d) \in \Pi_{m,M}} c_{\pi(d)} \pi(d) \quad \text{with} \quad c_{\pi(d)} \in \mathbb{Q} \quad \text{and} \quad \sum_{\pi(d) \in \Pi_{m,M}} c_{\pi(d)} = 1.$$

It follows that

$$(5) \quad e_i(N) = \beta_0 \cdot \sum_{\pi(d) \in \Pi_{m,M}} c_{\pi(d)} e_i(\pi(d))$$

Let $\prod_{k=1}^i (d_{j_k} - (j_k + k - 1))$ be one of the summands in $h_i(d)$. We claim that either $\prod_{k=1}^i (d_{j_k} - (j_k + k - 1)) = 0$, or else $d_{j_k} - (j_k + k - 1) > 0$ for $k = 1, \dots, i$. The claim will then imply that

$$(6) \quad e_i(\pi(d)) \leq e_i(\pi(d'))$$

whenever we have $d_i \leq d'_i$ for $i = 1, \dots, s$.

In order to prove the claim suppose that $\prod_{k=1}^i (d_{j_k} - (j_k + k - 1)) \neq 0$. Since $d_i \geq i$ for all i , we must then have that $d_{j_1} - j_1 > 0$. Assume that not all factors $d_{j_k} - (j_k + k - 1)$ are positive and let ℓ be the smallest integer with $d_{j_\ell} - (j_\ell + \ell - 1) < 0$. Then $\ell > 1$ and $d_{j_{\ell-1}} - (j_{\ell-1} + \ell - 2) > 0$. It follows that

$$d_{j_{\ell-1}} - (j_{\ell-1} + \ell - 2) - (d_{j_\ell} - (j_\ell + \ell - 1)) \geq 2,$$

equivalently

$$j_\ell - j_{\ell-1} \geq d_{j_\ell} - d_{j_{\ell-1}} + 1.$$

This is a contradiction, since $d_1 < d_2 < \dots < d_s$.

Now (5) and (6) imply that

$$\begin{aligned} e_i(\pi(m)) &\leq \min\{e_i(\pi(d)) : \pi(d) \in \Pi_{m,M}\} \\ &\leq \frac{e_i(N)}{\beta_0} \leq \max\{e_i(\pi(d)) : \pi(d) \in \Pi_{m,M}\} = e_i(\pi(M)), \end{aligned}$$

as desired. \square

Conjecture 2.2 is proved in several cases by Boij and Söderberg, and hence also proves our conjecture in these cases. We single out two such cases.

Corollary 2.4. *Let N be a Cohen-Macaulay S -module of codimension two, generated in degree 0, or let $N = S/I$ where I is a Gorenstein ideal of codimension 3. Then the bounds for the Hilbert coefficients given in Conjecture 2.1 hold.*

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